The computation of the Conformal Killing Vectors of an $1+(n-1)$ decomposable metric

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1 Introduction

A Conformal Killing Vector (CKV) ξ^a with conformal factor ψ of a general metric g_{ab} is defined by the equation[1]:

$$\mathcal{L}_\xi g_{ab} = \xi_{a;b} + \xi_{b;a} = 2\psi g_{ab} \tag{1}$$

where a semicolon denotes covariant differentiation with respect to the metric g_{ab}.

If $\psi = 0$ the CKV reduces to a Killing Vector (KV) and if $\psi =$const.$\neq 0$ the CKV reduces to a Homothetic Killing Vector (HKV). The determination of the CKVs of a general metric is important because, among others, it allows the simplification of the metric and consequently the easier study of the geometry of the space. For example if ξ^a is a non null gradient CKV i.e $\xi^a = \phi^{,a}$ then [1] there exists a coordinate system $\{x^1, x^\alpha\}$ such that $\phi^{,a} = \delta^a_1$, $\psi = \psi(x^1)$ and :

$$g_{ab} = g_{11}(dx^1)^2 + \frac{1}{g_{11}}\Gamma_{\alpha\beta}(x^2, x^3, .., x^n)dx^\alpha dx^\beta \tag{2}$$

where:

$$g_{11} = \frac{1}{2\int \psi(x^1)dx^1 + C} \tag{3}$$

and $\Gamma_{\alpha\beta}(x^2, x^3, .., x^n)$ are smooth functions on their arguments. A recent generalization of this result to non gradient non-null CKVs has appeared in [2].

As a roule, the determination of the CKVs of a metric is via the solution of the differential equations (1) which is not always an easy or possible task. Thus it is important to developed general theorems which for general classes of metrics will allow the computation of the CKVs without the explicit solution of the differential equations. In the present paper one such method is developed for the calculation of the CKVs of a general $1+(n-1)$ decomposable metric in terms of the Killing vectors and the gradient CKVs of the associated $(n-1)$ metric.

[1] Latin indices take the values $1, 2, ..., n$; Greek indices take the values $2, 3, ..., n$. The signature of the metric is arbitrary. Round and squared brackets enclosing indices denote symmetric and antisymmetric part respectively. Numbers in brackets refer to the references at the end of the paper.

2 The CKVs of a $1+(n-1)$ metric

A metric g_{ab} is called $1+(n-1)$ globally decomposable if it admits a non-null covariantly constant vector field M^a say i.e. $M_{a;b}=0$. It follows that M^a is a gradient KV hence Petrov's statement applies which means that there exist coordinates $\{x^1, x^\alpha\}$ in which $\mathbf{M}=\partial x^1$ and the metric is written:

$$ds^2=\epsilon(\mathbf{M})(dx^1)^2+g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta \tag{4}$$

$\epsilon(\mathbf{M})=M^2$ is the sign of M^a and the quantities $g_{\alpha\beta}(x^\gamma)$ are the components of the metric of the $(n-1)$ space x^1 =const.

Many well known and important metrics, especially in General Relativity, are globally decomposable or conformally related to a globally decomposable spacetime. For example such classes of metrics are the Friedmann-Robertson-Walker metric $ds^2=-dt^2+R^2(t)\left[\left(\frac{1}{1+\frac{\varepsilon}{4}\mathbf{x}^2}\right)^2(dx^2+dy^2+dz^2)\right]$, the Gödel-type metrics [5, 6] $ds^2=-[dt+H(r)d\Phi]^2+dr^2+D^2(r)d\Phi^2+dz^2$ which contain the well known Gödel metric etc. The main result of this paper is the following Theorem:

Theorem 2.1. *All proper CKVs X^a of an $1+(n-1)$ $(n\geq 3)$ metric g_{ab} are of the form*

$$X^a=f(x^a)\partial x^1+K^a \tag{5}$$

where $f(x^a)$ is a smooth function and K^α are CKVs of the $(n-1)$-metric $g_{\alpha\beta}$ of the form:

$$K^\alpha=\frac{1}{p}m(x^1)\xi^\alpha+L^\alpha(x^\beta) \tag{6}$$

such that:
(a) $\xi_\alpha=A_{,\alpha}$ is a gradient CKV of the $(n-1)$-metric $g_{\alpha\beta}$ whose conformal factor $\lambda(\xi)$ satisfies the relation

$$\lambda(\xi)_{|\alpha\beta}=p\lambda(\xi)g_{\alpha\beta} \tag{7}$$

p being a non-vanishing constant.
(b) the function $m(x^1)$ satisfies the equation $\overset{**}{m}+\epsilon pm=0$ where "" denotes differentiation with respect to the co-ordinate x^1 and ϵ is the sign of the gradient KV which decomposes the spacetime. Furthermore the function $f(x^a)$ is defined by the equation $f(x^a)=-\frac{\epsilon}{p}\overset{*}{m}\lambda(\xi)+Bx^1$.*
(c) L_α is a KV or a HKV of the $(n-1)$-metric $g_{\alpha\beta}$ which is not a gradient vector field and its bivector equals $F_{ab}(\mathbf{K})$. Then the non-gradient KVs of the $(n-1)$-metric are identical with those of the n-metric and the HKVs of the $1+(n-1)$ metric are of the form $Bx^1\partial_1+L^\alpha\partial_\alpha$ where L^α is HKV of the $(n-1)$-metric with conformal factor B. □

Proof

Suppose that X^a is a general (smooth) vector field of the $1+(n-1)$-metric g_{ab}. We decompose X^a along and normally to M^a:

$$X_a=f(x^b)M_a+X'_a \tag{8}$$

where $X'_a = h^b_a X_b$, $h_{ab} = g_{ab} - \epsilon(\mathbf{M})M_a M_b$ is the projection tensor and $f(x^a)$ is a smooth function. We define the vector K_α in the $(n-1)$-space x^1 =const. by the requirement $X'_a = K_\alpha \delta^\alpha_a$ so that:

$$X_a = f(x^b)\zeta_a + K_\alpha \delta^\alpha_a. \tag{9}$$

Taking the covariant derivative of the smooth vector field X^a it is always possible to write:

$$X_{a;b} = \psi(\mathbf{X})g_{ab} + H_{ab}(\mathbf{X}) + F_{ab}(\mathbf{X}) \tag{10}$$

where the quantities $\psi(\mathbf{X}), H_{ab}(\mathbf{X})$ and $F_{ab}(\mathbf{X})$ are defined as follows:

$$\begin{aligned}
\psi(\mathbf{X}) &= \frac{1}{n}X^a_{;a} \\
H_{ab}(\mathbf{X}) &= X_{(a;b)} - \psi(\mathbf{X})g_{ab} \\
F_{ab}(\mathbf{X}) &= X_{[a;b]}
\end{aligned} \tag{11}$$

Combining eqs (9) and (10) we obtain the following relations:

$$4\psi(\mathbf{X}) = \overset{*}{f} + 3\lambda(\mathbf{K}) \tag{12}$$

$$H_{ab}(\mathbf{X}) = \begin{pmatrix} \epsilon(\overset{*}{f} - \psi(\mathbf{X})) & \frac{\epsilon}{2}(f_{,\alpha} + \epsilon \overset{*}{K}_\alpha) \\ \frac{\epsilon}{2}(f_{,\alpha} + \epsilon \overset{*}{K}_\alpha) & \mathcal{H}_{\alpha\beta}(\mathbf{K}) + \frac{1}{4}(\lambda - \overset{*}{f})g_{\alpha\beta} \end{pmatrix} \tag{13}$$

$$F_{ab}(\mathbf{X}) = \begin{pmatrix} \bigcirc & \frac{\epsilon}{2}(f_{,\alpha} - \epsilon \overset{*}{K}_\alpha) \\ -\frac{\epsilon}{2}(f_{,\alpha} - \epsilon K_\alpha) & \mathcal{F}_{\alpha\beta}(\mathbf{K}) \end{pmatrix} \tag{14}$$

where the vector field K_α has been decomposed in a similar way:

$$K_{\alpha|\beta} = \lambda(\mathbf{K})g_{\alpha\beta} + \mathcal{H}_{\alpha\beta}(\mathbf{K}) + \mathcal{F}_{\alpha\beta}(\mathbf{K}) \tag{15}$$

and "|" denotes covariant differentiation w.r.t. $(n-1)$-metric $g_{\alpha\beta}$.

The requirement that X^a is a CKV $H_{ab}(\mathbf{X}) = 0$ implies:

$$\overset{*}{f} = \psi(\mathbf{X}) \tag{16}$$

$$f_{,\alpha} = -\epsilon \overset{*}{K}_\alpha \tag{17}$$

$$\mathcal{H}_{\alpha\beta}(\mathbf{K}) = 0 \qquad \text{and} \qquad \psi(\mathbf{X}) = \lambda(\mathbf{K}). \tag{18}$$

Taking into account the integrability conditions of f i.e. $\overset{*}{f}_{,\alpha} = (f_{,\alpha})^*$ and $f_{,\alpha\beta} = f_{,\beta\alpha}$ we obtain the additional equations:

$$\lambda(\mathbf{K})_{,\alpha} = -\epsilon \overset{**}{K}_{\alpha} \tag{19}$$

$$\overset{*}{\mathcal{F}}_{\alpha\beta} (\mathbf{K}) = 0. \tag{20}$$

Consider first the KVs of the n-metric $g_{\alpha\beta}$. These are defined by the condition $\psi(\mathbf{X}) = \lambda(\mathbf{K}) = 0$ which by (19) and (20) implies that $K_{\alpha}(x^i) = K_{\alpha}(x^{\rho})$ otherwise the n-metric admits further a covariantly constant vector field which we do not assume to be the case. Then (16) and (17) imply that f =const. and without loss of generality we may take $f = 0$. We conclude that the KVs of the $(n-1)$-metric $g_{\alpha\beta}$ are KVs of the full metric g_{ab}. Furthermore $F_{ab}(\mathbf{X}) \equiv \mathcal{F}_{\alpha\beta}\delta^{\alpha}_{a}\delta^{\beta}_{b}$.

Using the same arguments we prove that the HKVs of the n-metric g_{ab} are of the form:

$$Bx^1\partial_1 + L^{\alpha}\partial_{\alpha} \tag{21}$$

where L^{α} is HKV of the $(n-1)$-metric $g_{\alpha\beta}$ with conformal factor B. It is woth noticing that the full metric g_{ab} admits HKVs if and only if the $(n-1)$-metric $g_{\alpha\beta}$ admits HKVs.

We consider next the proper CKVs of the n-metric. Differentiating equation (19) we get:

$$\lambda(\mathbf{K})_{,\alpha}|_{\beta} = -\epsilon \overset{**}{\lambda} (\mathbf{K}) g_{\alpha\beta}. \tag{22}$$

The form of equation (22) and the decomposability of the n-metric implies that we must look for solutions of the form:

$$\lambda(\mathbf{K}) = m(x^1)A(x^{\rho}) + B(x^{\rho}). \tag{23}$$

Differentiating and using equation (19) we find:

$$m(x^1)A_{,\alpha} + B_{,\alpha} = -\epsilon \overset{**}{K}_{\alpha} . \tag{24}$$

Differentiating again and using equation (22) we get:

$$m(x^1)A_{|\alpha\beta} + B_{|\alpha\beta} = -\epsilon \overset{**}{m} (x^1)A(x^{\rho})g_{\alpha\beta}. \tag{25}$$

Because $A(x^{\rho}), B(x^{\rho})$ are functions of x^{ρ} only, equation (25) implies that:

$$m(x^1)A_{|\alpha\beta} + \epsilon \overset{**}{m} (x^1)A(x^{\rho})g_{\alpha\beta} = C_1 \tag{26}$$

$$B_{|\alpha\beta} = -C_1 \tag{27}$$

where C_1 is a constant.

Differentiating (26) w.r.t. x^1 we find:

$$A_{|\alpha\beta} = pA(x^{\rho})g_{\alpha\beta} \tag{28}$$

$$\overset{**}{m} + \epsilon p m = C_2 \tag{29}$$

where p, C_2 are constants. Equation (28) says that if $p \neq 0$ then $A_{,\alpha}$ is a gradient CKV of $g_{\alpha\beta}$ with conformal factor $pA(x^\rho)$ and if $p=0$ then $A_{,\alpha}$ is a gradient KV. Consequently we require $p \neq 0$. We compute easily $(A_{,\alpha}A^{,\alpha})_{|\beta} = 2pA_{,\beta}$ thus $A_{,\alpha}$ is not a null vector field. Combining (28) and (29) with (26) we find $C_1 = C_2 = 0$. Thus $B_{,\alpha} = 0$ and the function $m(x^1)$ satisfies the equation:

$$\overset{**}{m} + \epsilon p m = 0 \tag{30}$$

which means that ($p \neq 0$) $m(x^1) = \sin(\sqrt{p}x^1), \cos(\sqrt{p}x^1)$ for $\epsilon p = 1$ and $m(x^1) = \sinh(\sqrt{p}x^1), \cosh(\sqrt{p}x^1)$ for $\epsilon p = -1$.

Integrating (19) it follows:

$$\overset{*}{K}_\alpha = -\epsilon \int m(x^1)dx^1 A_{,\alpha} + D_\alpha(x^\beta). \tag{31}$$

Differentiating and taking the antisymmetric part we find that $D_\alpha(x^\beta)$ is a gradient vector field which we denote by $E_{,\alpha}(x^\beta)$.

Replacing (30) in (31) and integrating we find:

$$K_\alpha = \frac{1}{p} m(x^1)A_{,\alpha} + E_{,\alpha}x^1 + L_\alpha(x^\beta). \tag{32}$$

Differentiating (32) and using the fact that K_α is a CKV of $g_{\alpha\beta}$ with conformal factor $\lambda(\mathbf{K})$ we find:

$$E_{|\alpha\beta} = 0 \tag{33}$$

$$L_{\alpha|\beta} = Bg_{\alpha\beta} + F_{\alpha\beta}(\mathbf{K}). \tag{34}$$

From (33) we obtain $E_{,\alpha} = 0$. Thus the proper CKV K_α takes the form:

$$K_\alpha = \frac{1}{p} m(x^1)A_{,\alpha} + L_\alpha(x^\beta) \tag{35}$$

where the vector field L_α is a KV or a HKV of the $(n-1)$-metric whose bivector is equal to the bivector of K_α. Finally integrating eqn (16) we obtain:

$$f(x^a) = -\frac{\epsilon}{p} \overset{*}{m} \lambda(\xi) + Bx^1. \tag{36}$$

From Theorem 2.1 we conclude that for the computation of the proper CKVs of a $1+(n-1)$ decomposable space one has to know only the KVs, the HKV and the gradient CKVs of the $(n-1)$ space x^1 =const.. This reduction in the computation of the CKVs is essential and as we will show in the next section in many cases allows us to compute the CKVs without solving the partial differential equations (1).

One sattle point is the case $n=3$ because then the $(n-1)$ space is two dimentional and has an infinite dimentional conformal algebra. In this case we use only the gradient CKVs (28) of the two dimentional space which form a closed subalgebra.

Using Theorem 2.1 we can prove that a given metric g_{ab} does not admit *proper* CKVs. Indeed from Theorem 2.1 we infer that the metric g_{ab} does not admit proper CKVs if the $(n-1)$ space metric $g_{\alpha\beta}$ does not admit gradient CKVs. To find a necessary condition

for this to be the case we apply Ricci's identity to the vector defined in (35) and using eqs (23),(28) we obtain:

$$m(x^1)\left[\lambda(\xi)_{,\gamma}g_{\alpha\beta}-\lambda(\xi)_{,\beta}g_{\alpha\gamma}-\frac{1}{p}R_{\sigma\alpha\beta\gamma}\lambda(\xi)^{,\sigma}\right]=R_{\sigma\alpha\beta\gamma}L^{\sigma}-L_{\alpha|\beta\gamma}+L_{\alpha|\gamma\beta}. \tag{37}$$

The rhs vanishes identically due to Ricci's identity applied to the vector L^{α}. Thus this equation gives the condition:

$$\lambda(\xi)_{,\gamma}g_{\alpha\beta}-\lambda(\xi)_{,\beta}g_{\alpha\gamma}=\frac{1}{p}R_{\sigma\alpha\beta\gamma}\lambda(\xi)^{,\sigma}. \tag{38}$$

Contracting the indices α,γ we get:

$$\lambda(\xi)_{,\alpha}=-\frac{1}{2p}R_{\alpha\beta}\lambda(\xi)^{,\beta}. \tag{39}$$

This equation shows that $\lambda(\xi)_{,\alpha}$ is an eigenvector of the Ricci tensor associated with the $(n-1)$-metric $g_{\alpha\beta}$ with non-zero eigenvalue $-\frac{1}{2p}$. To express this statement in terms of the metric only we rewrite (39) as follows:

$$(R_{\alpha\beta}+2pg_{\alpha\beta})\lambda(\xi)^{,\beta}=0. \tag{40}$$

Thus in order that the $(n-1)$-metrtic $g_{\alpha\beta}$ does not admit proper gradient CKVs the following necessary (not sufficient) condition must be satisfied:

$$\det(R_{\alpha\beta}+2pg_{\alpha\beta})=0. \tag{41}$$

This condition can be applied directly to any given $1+(n-1)$-metric and together with (7) and (39) provides a simple criterion on the existence of gradient CKVs by the $(n-1)$-metric $g_{\alpha\beta}$ and, by Theorem 2.1., for the existense of proper CKVs for the $1+(n-1)$ metric g_{ab}.

3 Applications

One important class of applications of Theorem 2.1 exists when the $(n-1)$ space is a space of constant curvature. Indeed it is well known that the metric g_{ab} of a space of constant curvature is conformally related to the flat metric $\bar{g}_{ab}$:

$$g_{ab}=N^2(x^c)\bar{g}_{ab} \tag{42}$$

where $N(x^c)=\frac{1}{1+\frac{K}{4}(x^a x_a)}$ and $K=\frac{R}{12}$, R being the curvature scalar of the space.

This means that these metrics have the same CKVs but with different conformal factors and bivectors. If we denote by $\phi(\mathbf{X})$, $\bar{F}_{ab}(\mathbf{X})$ the conformal factor and the bivector of the generic CKV X^a for the flat metric and $\psi(\mathbf{X})$, $F_{ab}(\mathbf{X})$ the corresponding quantities for the metric g_{ab} then the following relations are easy to establish:

$$\psi(\mathbf{X})=\mathbf{X}(\ln N)+\phi(\mathbf{X}) \tag{43}$$

$$F_{ij}(\mathbf{X}) = N^2\bar{F}_{ij}(\mathbf{X}) - 2NN_{[,i}X_{j]}. \tag{44}$$

The generic CKV of the flat metric $\bar{g}_{ij}$ is:

$$X^i = a^i + a^i_{\cdot\cdot j}x^j + bx^i + 2(\mathbf{b}\cdot\mathbf{x})x^i - b^i(\mathbf{x}\cdot\mathbf{x}) \tag{45}$$

where a^i, $a^i_{\cdot\cdot j}$, b, b^i are constants and $(\mathbf{x}\cdot\mathbf{x}) = \overline{g}_{ij}x^ix^j$. KVs are defined by the constants a^i, $a^i_{\cdot\cdot j}$, there exists only one HKV defined by the constant b and the constants b^i define the remaining n Special CKVs[2].

Following standard notation [3] we write for the CKVs of the metric g_{ab}:

$$\begin{array}{lll} \mathbf{P}_i = \partial_i, \qquad \mathbf{M}_{ij} = x_i\partial_j - x_j\partial_i & & (n \text{ KVs}) \\ \mathbf{H} = x^i\partial_i & & (1 \text{ HKV}) \\ \mathbf{K}_i = 2x_i\mathbf{H} - (\mathbf{x}\cdot\mathbf{x})\mathbf{P}_i & & (n(n-1)/2 \text{ SCKVs}) \end{array}$$

with conformal factors

$$\begin{aligned} \psi(\mathbf{P}_i) &= -\frac{KN}{2}x_i, \psi(\mathbf{M}_{ij}) = 0 \\ \psi(\mathbf{H}) &= 1 - \frac{KN}{2}\mathbf{x}^2 \\ \psi(\mathbf{K}_i) &= 2Nx_i. \end{aligned} \tag{46}$$

and bivectors:

$$\begin{aligned} F_{ij}(\mathbf{P}_r) &= KNx_{[j}\delta^r_{i]} \\ F_{ij}(\mathbf{M}_{rs}) &= N^2\delta^{rs}_{ij} - KNx_{[j}a_{i]r}x^r \\ F_{ij}(\mathbf{H}) &= 0 \\ F_{ij}(\mathbf{K}_r) &= -4Nx_{[i}\delta^r_{j]} \end{aligned} \tag{47}$$

Instead of these vectors we consider the following basis of CKVs of the metric g_{ab}:

$$\mathbf{P}_i + \frac{K}{4}\mathbf{K}_i,\ \mathbf{M}_{ij} \tag{48}$$

$$\mathbf{H}, \mathbf{P}_i - \frac{K}{4}\mathbf{K}_i \tag{49}$$

[2] A CKV is called Special Conformal Killing Vector iff the conformal factor ψ satisfies the condition $\psi_{;ab} = 0$. The proper CKVs of flat metrics are all Special CKVs.

From (46)-(49) it follows that the vectors $\mathbf{P}_i+\frac{K}{4}\mathbf{K}_i, \mathbf{M}_{ij}$ are non-gradient KVs of the metric g_{ab} and that the vectors $\mathbf{H}, \mathbf{P}_i-\frac{K}{4}\mathbf{K}_i$ are gradient proper CKVs of g_{ab} with conformal factors:

$$\psi(\mathbf{H}) = 1-\frac{1}{2}KN\mathbf{x}^2 \tag{50}$$

$$\psi(\mathbf{P}_i-\frac{K}{4}\mathbf{K}_i) = -KNx_i. \tag{51}$$

Thus we have computed all the KVs and the proper gradient CKVs of any metric of constant curvature. Using this and Theorem 2.1 we can compute the conformal algebra of any $1+(n-1)$ space whose $(n-1)$ space is a space of constant curvature.

It is worth noticing that any such space is conformally flat. The proof is simple and has as follows. We have shown that the $(n-1)$ space admits $(n-1)+1=n$ gradient CKVs each of which gives $2n$ CKVs (via the two solutions $m(x^1)$) of the decomposable metric. To these vectors we add the $n+\frac{n(n-1)}{2}=\frac{n(n+1)}{2}$ KVs of the $(n-1)$ metric and we have in total $\frac{(n+1)(n+2)}{2}$ CKVs for the n-metric g_{ab}which means that this metric is conformally flat [4].

Let us show how the above apply to an important spacetime found by Reboucas and Texeira [6] called the ART (anti Reboucas-Tiomno) metric. This metric is defined as follows:

$$ds^2_{ART} = dz^2-\cos^2(r/a)d\tau^2+dr^2+\sin^2(r/a)d\phi^2. \tag{52}$$

By means of the transformation:

$$\begin{aligned}
\tilde{t} &= 2\frac{\sinh(\frac{\tau}{a})\cos\left(\frac{r}{a}\right)}{1-\cosh\left(\frac{\tau}{a}\right)\cos\left(\frac{r}{a}\right)}\\
\tilde{x} &= 2\frac{\sin\left(\frac{r}{a}\right)\cos(\frac{\phi}{a})}{1-\cosh\left(\frac{\tau}{a}\right)\cos\left(\frac{r}{a}\right)}\\
\tilde{y} &= 2\frac{\sin\left(\frac{r}{a}\right)\sin\left(\frac{\phi}{a}\right)}{1-\cosh\left(\frac{\tau}{a}\right)\cos\left(\frac{r}{a}\right)}
\end{aligned} \tag{53}$$

it takes the standard form:

$$ds^2_{ART} = dz^2+\frac{a^2}{\left[1+\frac{1}{4}(\tilde{x}^2+\tilde{y}^2-\tilde{t}^2\right]^2}\left(-d\tilde{t}^2+d\tilde{x}^2+d\tilde{y}^2\right)$$

i.e. it is a 1+3 metric.

It is easy to show that the 3-space $z=$const. has constant positive curvature $R=6/a^2$ hence Theorem 2.1 applies. Using the results above and transformation (53) we compute first the KVs of the 3-metric to be [6]:

$$
\begin{aligned}
\xi_1 &= \tan(\frac{r}{a})\cos(\frac{\phi}{a})\cosh(\frac{\tau}{a})\partial_\tau + \cos(\frac{\phi}{a})\sinh(\frac{\tau}{a})\partial_r - \cot(\frac{r}{a})\sin(\frac{\phi}{a})\sinh(\frac{\tau}{a})\partial_\phi \\
\xi_2 &= \tan(\frac{r}{a})\sin(\frac{\phi}{a})\cosh(\frac{\tau}{a})\partial_\tau + \sin(\frac{\phi}{a})\sinh(\frac{\tau}{a})\partial_r + \cot(\frac{r}{a})\cos(\frac{\phi}{a})\sinh(\frac{\tau}{a})\partial_\phi \\
\xi_3 &= \tan(\frac{r}{a})\cos(\frac{\phi}{a})\sinh(\frac{\tau}{a})\partial_\tau + \cos(\frac{\phi}{a})\cosh(\frac{\tau}{a})\partial_r - \cot(\frac{r}{a})\sin(\frac{\phi}{a})\cosh(\frac{\tau}{a})\partial_\phi \\
\xi_4 &= \tan(\frac{r}{a})\sin(\frac{\phi}{a})\sinh(\frac{\tau}{a})\partial_\tau + \sin(\frac{\phi}{a})\cosh(\frac{\tau}{a})\partial_r + \cot(\frac{r}{a})\cos(\frac{\phi}{a})\cosh(\frac{\tau}{a})\partial_\phi \\
\xi_5 &= \partial_\tau \qquad \xi_6 = \partial_\phi \qquad \xi_7 = \partial_z
\end{aligned} \tag{54}
$$

In order to determine the four gradient CKVs first we compute the gradient CKV $\mathbf{H}$. Using (50) and the transformation equations (53) we find:

$$
\begin{aligned}
\mathbf{H} &= a\left[\frac{\sinh(\frac{\tau}{a})}{\cos(\frac{r}{a})}\partial_\tau + \cosh(\frac{\tau}{a})\sin(\frac{r}{a})\partial_r\right] \\
\psi(\mathbf{H}) &= -\cos(\frac{r}{a})\cosh(\frac{\tau}{a})
\end{aligned} \tag{55}
$$

The rest three gradient CKVs can be determined by taking the commutators of $\mathbf{H}$ with the KVs (54) (we recall that the KVs of the 3-metric are identical with those of the 4-metric).

Finally, using Theorem 2.1., we find the following eight proper CKVs for the ART spacetime $(k = 1, 2, 3, 4)$:

$$
\begin{aligned}
\xi_{(k)\alpha} &= -a^2 A_{k,\alpha} \qquad \xi_{(k)3} = a^2 A_{k,3} \\
\psi\left[\xi_{(k)}\right] &= A_k
\end{aligned} \tag{56}
$$

$$
\begin{aligned}
\xi_{(k+4)\alpha} &= -a^2 B_{k,\alpha} \qquad \xi_{(k+4)3} = a^2 B_{k,3} \\
\psi\left[\xi_{(k+4)}\right] &= B_k
\end{aligned} \tag{57}
$$

where:

$$
\begin{aligned}
A_k &= \cos(\frac{r}{a})\left\{\cosh(\frac{\tau}{a})\sinh(\frac{z}{a}), \cosh(\frac{\tau}{a})\cosh(\frac{z}{a}), \sinh(\frac{\tau}{a})\sinh(\frac{z}{a}), \sinh(\frac{\tau}{a})\cosh(\frac{z}{a})\right\} \\
B_k &= \sin(\frac{r}{a})\left\{\cos(\frac{\phi}{a})\sinh(\frac{z}{a}), \cos(\frac{\phi}{a})\cosh(\frac{z}{a}), \sinh(\frac{\phi}{a})\sinh(\frac{z}{a}), \sinh(\frac{\phi}{a})\cosh(\frac{z}{a})\right\}.
\end{aligned} \tag{58}
$$

As a second application of Theorem 2.1. we prove that the Gödel metric does not admit CKVs. It is well known [7] that the Gödel metric admits five KVs and does not admit HKVs [8] but it does or does not admits proper CKVs. The answer is that the Gödel metric does not admit proper CKVs. To prove this we use the relation (39) and the necessary condition (41).

In Cartesian coordinates the Gödel metric is given by the following line element [9]:

$$ds^2 = -dt^2 - 2e^{ax}dtdy + dx^2 - \frac{1}{2}e^{2ax}dy^2 + dz^2 \tag{59}$$

where a is an arbitrary constant. Thus the Gödel metric is a 1+3 metric along the spacelike direction z and (41) applies. The Ricci tensor for the 3-metric $g_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 - 2e^{ax}dtdy + dx^2 - \frac{1}{2}e^{2ax}dy^2$ is calculated to be:

$$R_{\alpha\beta} = a^2dt^2 + 2a^2e^{ax}dtdy + a^2e^{2ax}dy^2. \tag{60}$$

hence the determinant $\det(R_{\alpha\beta} + 2pg_{\alpha\beta}) = 2p^2e^{2ax}(a^2 - 2p)$. The determinant vanishes when $p = a^2/2 > 0$. For this value of p the eigenvalue equation (39) gives easily that $\lambda(\xi)_{,x} = \lambda(\xi)_{,y} = \lambda(\xi)_{,t} = 0$, i.e. $\lambda(\xi)$ =const. which is a contradiction. Thus the Gödel space-time does not admit proper CKVs.

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